ANALYTICAL SOLUTION OF DYNAMIC FLEXURAL RESPONCE OF SYMMETRIC COMPOSITE TIMOSHENKO BEAMS UNDER HARMONIC FORCES

Mohammed Ali Hjaji, Hasan M. Nagiar and Ezedine G. Allaboudi

Mechanical and Industrial Engineering Department
University of Tripoli, Libya
Emails: m.hjaji@uot.edu.ly

الملخص

تمت دراسة الاستجابة الديناميكية العرضية للعتبات (الكمرات) ذات طبقات مركبة متماثلة ومعرضة لقوى عرضية توافقية مختلفة. تم اشتقاق معادلات الحركة الديناميكية وكذلك الشروط الحدية ذات العلاقة وفقا لنظرية تشوه القص من الدرجة الاولى وذلك باستخدام مبدأ هاملتون. تأثيرات تشوه القص وقصور الدوران ونسبة بويسون وكذلك اتجاه الألياف بالطبقات المركبة تم ادراجها بصيغ المعادلات الحالية، حيث تم الحصول على الحل المضبوط (closed form solution) للمعادلات الناتجة والحاكمة للاستجابة الديناميكية العرضية للحالة المستقرة لعتبات توميشنكو (Timoshenko beams) ذات طبقات مركبة متماثلة الألياف. كذلك تم إيجاد صيغ الحل المضبوط للإهتزازات الديناميكية العرضية المستقرة لعتبات ذات تثبيت بسيط (simply supported) وأخرى ذات تثبيت كابولي (cantilever). تم إستعراض بعض الأمثلة لتوضيح إمكانية تطبيق صيغ الحل المضبوط للعديد من الأحمال العرضية التوافقية لعتبات ذات مركبات متماثلة الطبقات وزوايا الألياف. أظهرت النتائج المتحصل عليها من هذه الدراسة صحة ودقة الحل الحالي (present solution) وذلك من خلال مقارنتها بنتائج مماثلة منشورة باستخدام طريقة العناصر المتناهية (finite element) واخرى باستخدام الحل الدقيق (Exact solution).

ABSTRACT

The flexural dynamic response of symmetric laminated composite beams subjected to general transverse harmonic forces is investigated. The dynamic equations of motion and associated boundary conditions based on the first order shear deformation are derived through the use of Hamilton's principle. The influences of shear deformation, rotary inertia, Poisson's ratio and fibre orientation are incorporated in the present formulation. The resulting governing flexural equations for symmetric composite Timoshenko beams are solved exactly and the closed form solutions for steady state flexural response are then obtained for cantilever and simply supported boundary conditions. The applicability of the analytical closed-form solution is demonstrated via several examples with various transverse harmonic loads and symmetric cross-ply and angle-ply laminates. Results based on the present solution are assessed and validated against other well established finite element and exact solutions available in the literature.

KEYWORDS: Analytical Solution; Flexural Response; Harmonic Forces; Symmetric Laminated Beams; Steady State Response.

INTRODUCTION

Composite laminated beams are among the most important structural components widely used in aircraft wing and fuselage structures, helicopter blades, vehicle axles, propellant and turbine blades, ship and marine structural frames due to their excellent features such as high strength-to-weight and stiffness-to-weight ratios. In these applications, composite laminated beams are frequently subjected to cyclic dynamic loading (e.g., harmonic excitations). Sources of such forces include aerodynamic effects, hydro-dynamic wave motion and wind loading. Also, harmonic forces may arise from unbalance in rotating machinery and propellants and reciprocating machines. In such applications, composite beams under harmonic forces cause an undesirable vibrations and prone to fatigue failures, an important limit state when designing these composite laminated beams. Under harmonic forces, the transient component of response which is induced only at the beginning of the excitation tends to dampen out quickly and is thus of no importance in assessing the fatigue life of a composite beam. In contrast, the steady state component of the response is sustained for a long time and is thus of particular importance in fatigue design and is the subject of the present study. Within this context, the aim of this study is to develop an accurate and efficient solution, which captures and isolates the steady state response. The present analytical closed form solution is also able to capture the quasi-static response and predict the eigen-frequencies and eigen-modes of the composite laminated beam.

While the dynamic analysis of composite laminated beams based on different beam theories was the subject of significant research studies during the past few years, but most of these studies are restricted to free vibrations of composite laminated beams. Many researchers developed and presented the analytical exact solutions and finite element techniques for free vibration response of composite laminated beams. Among them, [1] developed an exact solution based on higher-order shear deformation theory to study the free vibration behavior of cross-ply rectangular beams with arbitrary boundary conditions. Based on the transfer matrix method, [2] investigated the in-plane and out-of-plane free vibration problem of symmetric cross-ply laminated beams. References [3-4] presented the exact expressions for the frequency equation and mode shapes for composite Timoshenko cantilever beams. His formulation captured the effects of material coupling between bending and torsional modes, shear deformation and rotary inertia. Reference [5] studies the free vibration of composite laminated beams using a higher-order shear deformation theory. The differential quadrature method is used to obtain the numerical solution of the governing differential equations for symmetrically and anti-symmetrically composite beams with rectangular cross-section and for various boundary conditions. Reference [6] presented a displacement based layerwise beam theory and applied it to laminated ($0^o/90^o$) and ($0^o/90^o/0^o$) beams subjected to sinusoidal load. References [7-9] developed the exact dynamic stiffness matrix method free vibration analyses of arbitrary laminated composite beams based on first order shear deformation, trigonometric shear deformation and higher-order shear deformation beam theories. The effects of shear deformation, rotary inertia, Poisson's ratio, axial force and extensional-bending coupling deformations are considered in their mathematical formulations. Recently, [10] developed a two-noded C_1 finite beam element with five degrees of freedom per node to study the free vibration and buckling analyses of composite cross-ply laminated beams by using the refined shear deformation theory. Their formulations account for the parabolical variation of the shear strains through the beam depth and all coupling coming from the material anisotropy. More recently, [11] developed a finite element model based on the first order shear deformation theory to predict the static and free vibration analyses for isotropic and orthotropic beams with different boundary conditions and length-to-thickness ratios.

It should be remarked that the previous studies are mainly focused to free vibration analysis, with no attention on studying the dynamic analysis of composite laminated beams subjected to harmonic forces. To the best of author's knowledge, no study in the literature reported an analytical closed-form solution for dynamic flexural response of composite symmetric laminated Timoshenko beams under harmonic forces. Then, the purpose of this paper is to formulate the governing field equations and boundary conditions for the problem and provide the closed form exact solutions for symmetric laminated beams of rectangular cross-sections subjected to various transverse harmonic excitations. The present analytical closed form solution captures the effects of shear deformation, rotary inertia, Poisson's ratio and fibre orientation on quasi-static and steady state dynamic responses. The present general analytical solution is (i) appropriate and efficient in analyzing the forced bending vibration of composite laminated beams subjected to transverse harmonic excitations, (ii) suitable to achieve simple preliminary design considerations of composite beams and (iii) used as benchmarks for checking the accuracy of the results obtained from the numerical or approximate solutions.

MATHEMATICAL FORMULATION

Kinematics Relations

A prismatic multilayered beam with length L, thickness h and width b, as shown in Figure (1), is considered. In the right-handed Cartesian coordinate system (X,Y,Z) defined on the mid-plane of the beam, the X axis is coincident with the beam axis, Y and Z are coincident with the principal axes of the cross-section. Since the cross-section of the composite beam has two axes of symmetry (i.e., Y and Z axes), therefore, no coupling exists between bending and torsion responses, i.e., the present study is restricted to flexural behavior in the $X-Z$ plane. Thus, the displacements for a general point $p(x,z)$ of height z from the centroidal axis of beam based on the first order shear deformation beam theory are assumed to take the form:

$$u_p(x,z,t)=u(x,t)+z\phi_x(x,t) \quad (1)$$

$$v_p(x,z,t)=0 \quad (2)$$

$$w_p(x,z,t)=w(x,t) \quad (3)$$

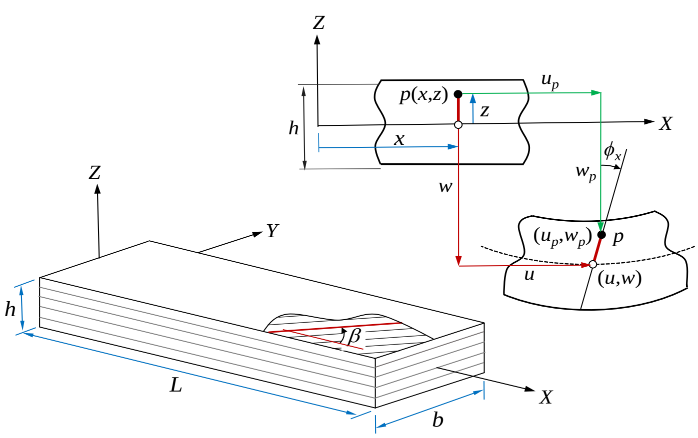

Figure 1: Coordinate system and displacements

in which $u(x,t)$ and $w(x,t)$ are the axial and transverse displacements of a point $p(x,z)$ on the mid-plane in the X and Z directions, $v_p(x,z,t)$ is the lateral displacement, and $\phi_x(x,t)$ is the rotation of the normal to the mid-plane about Y axis, where x and t are spanwise coordinate and time, respectively.

Strain-Displacement Relations

The strain relations of the beam associated with the small-displacement theory of elasticity are given as:

$$\varepsilon_{xx} \approx \frac{\partial u_p}{\partial x} = \bar{\varepsilon}_{xx} + z\kappa_x \quad , \text{ and } \qquad \gamma_{xz} \approx \frac{\partial w_p}{\partial x} + \phi_x \tag{4}$$

where $\bar{\varepsilon}_{xx} = \partial u / \partial x = u'$ is the mid-plane axial strain, $\kappa_x = \partial \phi_x / \partial x = \phi'_x$ is the bending curvature, and the primes denote the differentiation with respect to x.

Constitutive Equations of Symmetric Laminated Beams

The constitutive equations for a symmetric laminated beam (in which the extensional-bending coupling coefficients $B_{ij} = 0$ for $i, j = 1,2,6$) based on the first order shear deformation theory can be obtained by using the classical lamination theory to give:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & & & \\ A_{12} & A_{22} & A_{26} & & 0 & \\ A_{16} & A_{26} & A_{66} & & & \\ & & & D_{11} & D_{12} & D_{16} \\ & 0 & & D_{12} & D_{22} & D_{26} \\ & & & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\gamma}_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{5}$$

where N_x, N_y are the normal forces, and N_{xy} is the in-plane force, while M_x, M_y and M_{xy} are the bending and twisting moments, $\bar{\varepsilon}_{xx}$, $\bar{\varepsilon}_{yy}$ and $\bar{\gamma}_{xy}$ are normal and shear strains, κ_x, κ_y and κ_{xy} are the bending and twisting curvatures, A_{ij} and D_{ij} denote the extensional and bending stiffnesses, respectively, and are expressed as functions of laminate ply orientation and material properties:

$$A_{ij}, D_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij} \left[1, z^2\right] dz \;, \quad (\text{for } i, j = 1,2,6) \tag{6}$$

where $\bar{Q}_{ij}$ are the transformed reduced stiffnesses and are given by the expressions [12]:

$$\bar{Q}_{11} = Q_{11}\cos^4\beta + 2(Q_{12} + 2Q_{66})\sin^2\beta\cos^2\beta + Q_{22}\sin^4\beta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\beta\cos^2\beta + Q_{12}\left(\sin^4\beta + \cos^4\beta\right)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\beta + 2(Q_{12} + 2Q_{66})\sin^2\beta\cos^2\beta + Q_{22}\cos^4\beta$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\beta\cos^3\beta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\beta\cos\beta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\beta\cos\beta + (Q_{12} - Q_{22} + 2Q_{66})\sin\beta\cos^3\beta$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\beta\cos^2\beta + Q_{66}\left(\sin^4\beta + \cos^4\beta\right)$$

in which β is the angle between the fiber direction and longitudinal axis of the beam Figure (1), Q_{11}, Q_{12}, Q_{22} and Q_{66} are the stiffness constants and are given in terms of engineering elastic constants by:

$$Q_{11} = E_{11}/(1 - \upsilon_{12}\upsilon_{21}), \quad Q_{12} = \upsilon_{12}E_{11}/(1 - \upsilon_{12}\upsilon_{21}) = \upsilon_{21}E_{22}/(1 - \upsilon_{12}\upsilon_{21})$$

$Q_{22} = E_{22}/(1 - \upsilon_{12}\upsilon_{21})$, and $Q_{66} = G_{12}$.

The present formulation captures the effect of transverse shear deformation due to bending [10], then:

$$Q_{xz} = A_{55}\gamma_{xz} = A_{55}\left(\frac{\partial w}{\partial x} + \phi_x\right) = A_{55}(w' + \phi_x) \tag{7}$$

in which Q_{xz} is the transverse shear force per unit length, $A_{55} = k\int_{-h/2}^{h/2}\bar{Q}_{55}dz$, where $\bar{Q}_{55} = G_{13}\cos^2\beta + G_{23}\sin^2\beta$, k is the correlation shear factor and is taken as 5/6 to account for the parabolic variation of the transverse shear stresses, the constants E_{11}, E_{22} are Young moduli, G_{12}, G_{13}, G_{23} are shear moduli, and υ_{12}, υ_{21} are Poison ratios measured in the principal axes of the layer.

The laminated composite beam is subjected to flexural forces. Then, the lateral in-plane forces and moments in Y direction are negligible and set to zero, i.e., $N_y = N_{xy} = M_y = M_{xy} = 0$. In order to account for Poisson's ratios, the mid-plane strains $\bar{\varepsilon}_{yy}, \bar{\gamma}_{xy}$ and curvatures κ_y, κ_{xy} are assumed to be nonzero. For symmetric laminated beams, the extensional response is uncoupled from the flexural response of the beam, i.e., the bending stiffness coefficients B_{ij} are ignored ($B_{ij} = 0$ for $i = j = 1,2,6$), Thus, equation (5) is written as:

$$\begin{Bmatrix} N_x \\ M_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \kappa_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} u' \\ \phi'_x \end{Bmatrix} \tag{8}$$

where $\bar{A}_{11} = A_{11} + \left[\dfrac{A_{12}(A_{66} - A_{26}) + A_{16}(A_{22} - A_{26})}{\left(A_{26}^2 - A_{22}A_{66}\right)}\right]$, and

$$\bar{D}_{11} = D_{11} + \left[\frac{D_{12}(D_{66} - D_{26}) + D_{16}(D_{22} - D_{26})}{\left(D_{26}^2 - D_{22}D_{66}\right)}\right].$$

If Poisson's ratio effect is ignored, the coefficients $\bar{A}_{11}, \bar{D}_{11}$ in equation (8) are then replaced by the laminate stiffness coefficients A_{11}, D_{11}, respectively.

ENERGY EXPRESSIONS

The total kinetic energy T for symmetric laminated composite beam (i.e., both geometric and material symmetry with respect to the mid-surface) is given by:

$$T=\frac{1}{2}\int_0^L\int_{-h/2}^{h/2}\rho\left[\dot{u}_p^2+\dot{w}_p^2\right]bdzdx=\frac{1}{2}\int_0^L\left[I_1\dot{u}^2+I_2\dot{\phi}_x^2+I_1\dot{w}^2\right]bdx \tag{9}$$

where the dot denotes the derivative with respect to time, the densities I_1 and I_2 of the composite beam are introduced by:

$$I_1,I_2=\int_{-h/2}^{h/2}\rho\left[1,z^2\right]dz=\sum_{k=1}^{m}\rho_k\left[\left(z_k-z_{k-1}\right),\left(z_k^3-z_{k-1}^3\right)/3\right]$$

where ρ_k is the mas density of the k^{th} layer.

The total strain energy U for symmetric laminated composite beam is given by:

$$U=\frac{1}{2}\int_0^L\left[N_x\varepsilon_{xx_o}+M_x\kappa_x+Q_{xz}\gamma_{xz}\right]bdx$$

Substitution from equations (6) and (8) into the above equation, yields:

$$U=\frac{1}{2}\int_0^L\left[\bar{A}_{11}u'^2+\bar{D}_{11}\phi_x'^2+A_{55}\left(w'^2+2w'\phi_x+\phi_x^2\right)\right]bdx \tag{10}$$

The work done V by the applied harmonic axial and flexural forces can be written as:

$$\begin{aligned}V=&\int_0^L\left[q_z(x,t)w(x,t)+q_x(x,t)u(x,t)+m_x(x,t)\phi_x(x,t)\right]dx+\left[P_x(x_e,t)u(x_e,t)\right]_0^L\\&+\left[P_z(x_e,t)w(x_e,t)\right]_0^L+\left[M_x(x_e,t)\phi_x(x_e,t)\right]_0^L\end{aligned} \tag{11}$$

Expressions for Force Functions

The laminated composite beam illustrated in Figure (2) is assumed to be subjected to distributed harmonic forces and moments within the beam:

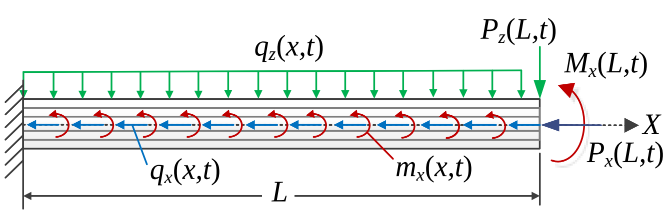

Figure 2: Composite cantilever under transverse harmonic forces and moments

$$q_z(x,t),q_x(x,t),m_x(x,t)=\left[\bar{q}_z(x),\bar{q}_x(x),\bar{m}_x(x)\right]e^{i\Omega t} \tag{12}$$

and to concentrated harmonic forces and moments at beam both ends:

$$P_x(x,t),P_z(x,t),M_x(x,t)=\left[\bar{P}_x(x),\bar{P}_z(x),\bar{M}_x(x)\right]_0^L e^{i\Omega t} \tag{13}$$

where Ω is the circular exciting frequency of the applied forces, $i=\sqrt{-1}$ is the imaginary constant, $q_z(x,t)$, $q_x(x,t)$ are the distributed axial and transverse harmonic forces, $m_x(x,t)$ is the distributed harmonic bending moment, $P_x(x,t)$, $P_z(x,t)$ are the concentrated axial and transverse harmonic forces, $M_x(x,t)$ is the concentrated harmonic bending moment, all forces and moments are applied at beam ends (i.e., $x=0,L$).

Steady State Displacement Functions

Under the given applied harmonic forces and moments, the displacement functions corresponding to the steady state component of the response are assumed to take the form:

$$u(x,t), w(x,t), \phi_x(x,t) = \left[U(x), W(x), \Phi_x(x)\right]e^{i\Omega t} \tag{14}$$

in which $U(x)$, $W(x)$ and $\Phi_x(x)$ are the amplitudes for axial translation, bending displacement, related bending rotation, respectively. Since the present formulation is intended to capture only the steady state response of the system, the displacement fields postulated in equation (14) neglect the transient component of the response.

HAMILTON’S VARIATIONAL PRINCIPLE

The dynamic differential equations of motion for laminated composite beam subjected to harmonic forces can be derived using Hamilton’s principle, which can be written as:

$$\int_{t_1}^{t_2} \delta\left(T - U + V\right)dt = 0, \qquad \text{where } \delta u = \delta w = \delta\phi_x = 0 \text{ at } t = t_1 \text{ and } t_2 \tag{15}$$

where t_1 and t_2 are two arbitrary time variables and δ denotes the first variation. From equations (12-14) and by substituting into the energy expressions (9-11), then, the resulting equations are substituted into Hamilton’s principle (15), performing integration by parts, the following governing equations of motion are obtained:

$$-\bar{A}_{11}U''(x) - I_1\Omega^2 U(x) = \bar{q}_x(x)/b \tag{16}$$

$$-I_1\Omega^2 W(x) - A_{55}W''(x) - A_{55}\Phi'_x(x) = \bar{q}_z(x)/b \tag{17}$$

$$A_{55}W'(x) - \bar{D}_{11}\Phi''_x(x) + \left(A_{55} - I_2\Omega^2\right)\Phi_x(x) = \bar{m}_x(x)/b \tag{18}$$

The related boundary conditions arising from the variational principle are:

$$\left[b\bar{A}_{11}U'(x)\right]\delta U(x)\Big|_0^L = \bar{P}_x(x)\Big|_0^L \tag{19}$$

$$bA_{55}\left[W'(x) + \Phi_x(x)\right]\delta W(x)\Big|_0^L = \bar{P}_z(x)\Big|_0^L \tag{20}$$

$$\left[b\bar{D}_{11}\Phi'_x(x)\right]\delta\Phi_x(x)\Big|_0^L = \bar{M}_x(x)\Big|_0^L \tag{21}$$

Equation (16) governs the longitudinal deformation of the symmetric laminated composite beam which is uncoupled with the remaining equations and can be solved independently. Equations (17-18) with related boundary conditions in (20-21) provide the flexural vibration and related rotation for symmetric laminated beams. The present study is focused on the analytical solution for the steady state dynamic response governed by the flexural equations.

ANALYTICAL CLOSED-FORM SOLUTION FOR FLEXURAL RESPONSE

Homogeneous Solution

The homogeneous solution of the flexural equations (17-18) is obtained by setting the right-hand side of the equations to zero, i.e. $\bar{q}_z(x)=\bar{m}_x(x)=0$. The homogeneous solution of the displacements is then assumed to take the exponential form:

$$\{\chi_h(x)\}_{2\times1}=\begin{Bmatrix} W_h(x) \\ \Phi_{x_h}(x) \end{Bmatrix}_{2\times1}=\begin{Bmatrix} c_{1,i} \\ c_{2,i} \end{Bmatrix}_{2\times1} e^{m_i x},\ \text{for } i=1,2,3,4 \tag{22}$$

where $\langle\chi_h(x)\rangle_{1\times2}=\langle W_h(x)\ \ \Phi_{x_h}(x)\rangle_{1\times2}$ is the vector of flexural displacement and rotation, and $\langle C\rangle_{1\times2}=\langle c_{1,i}\ \ \ c_{2,i}\rangle_{1\times2}$ is the vector of unknown integration constants. From equation (22), by substituting into equations (17-18), yields:

$$\begin{bmatrix} -\left(I_1\Omega^2+A_{55}m_i^2\right) & -A_{55}m_i \\ A_{55}m_i & \left(A_{55}-I_2\Omega^2-\bar{D}_{11}m_i^2\right) \end{bmatrix}_{2\times2}\begin{bmatrix} e^{m_i x} & 0 \\ 0 & e^{m_i x} \end{bmatrix}_{2\times2}\begin{Bmatrix} c_{1,i} \\ c_{2,i} \end{Bmatrix}_{2\times1}=\{0\}_{2\times1} \tag{23}$$

For a non-trivial solution, the determinant of the bracketed matrix in equation (23) is set to vanish, leading to the quartic equation of the form:

$$A_{55}\bar{D}_{11}\,m_i^4+\Omega^2(A_{55}I_2+I_1\bar{D}_{11})\,m_i^2+\Omega^2 I_1(I_2\Omega^2-A_{55})=0$$

which is depend upon section properties, material constants and exciting frequency. The above equation has the following four distinct roots;

$$m_{1,2}=\pm\sqrt{\frac{1}{2A_{55}\bar{D}_{11}}\left[-\Omega^2(A_{55}I_2+\bar{D}_{11}I_1)+\sqrt{\Omega\left[4A_{55}^2\bar{D}_{11}I_1+\Omega^2(A_{55}I_2-\bar{D}_{11}I_1)^2\right]}\right]},\ \text{and}$$

$$m_{3,4}=\pm i\sqrt{\frac{1}{2A_{55}\bar{D}_{11}}\left[\Omega^2(A_{55}I_2+\bar{D}_{11}I_1)+\sqrt{\Omega\left[4A_{55}^2\bar{D}_{11}I_1+\Omega^2(A_{55}I_2-\bar{D}_{11}I_1)^2\right]}\right]}$$

For each root m_i, there corresponds a set of unknown constants $\langle C\rangle_{i,1\times2}=\langle c_{1,i}\ \ c_{2,i}\rangle_{i,1\times2}$. By back-substitution into equation (23), one can relate constants $c_{1,i}$ to $c_{2,i}$ through $c_{1,i}=G_i c_{2,i}$, where $G_i=-A_{55}m_i\Big/\left[I_1\Omega^2+A_{55}m_i^2\right]$, for $i=1,2,3,4$.

The homogeneous solutions for the flexural displacement and $W_h(x)$ related bending rotation $\Phi_{x_h}(x)$ are obtained as:

$$\{\chi_h(x)\}_{1\times2}=\left[\bar{G}\right]_{2\times4}\left[E(x)\right]_{4\times4}\{\bar{C}\}_{4\times1} \tag{24}$$

where $\left[\bar{G}\right]_{2\times4}=\left[\begin{Bmatrix}G_1\\1\end{Bmatrix}\begin{Bmatrix}G_2\\1\end{Bmatrix}\begin{Bmatrix}G_3\\1\end{Bmatrix}\begin{Bmatrix}G_4\\1\end{Bmatrix}\right]_{2\times4}$, $\left[E(x)\right]_{4\times4}$ is a diagonal matrix consisting of the exponential functions $e^{m_i x}$ (for $i=1,2,3,4$), the vector of unknown integration constants $\langle\bar{C}\rangle_{1\times4}=\langle c_{2,1}\;\; c_{2,2}\;\; c_{2,3}\;\; c_{2,4}\rangle_{1\times4}$ is to be determined from the boundary conditions of the problem.

Particular Solution for Uniform Load Distribution

For a composite laminated beam subjected to distributed transverse forces and moments $\left(\bar{q}_z(x),\bar{m}_x(x)\right)e^{i\Omega t}=\left(\bar{q}_z,\bar{m}_x\right)e^{i\Omega t}$, the corresponding particular solution of the coupled bending equations (17-18) is given as:

$$\langle\chi_P\rangle_{1\times2}=\langle W_p\quad \Phi_{x_p}\rangle_{1\times2}=\left\langle\frac{-\bar{q}_z}{bI_1\Omega^2}\quad \frac{\bar{m}_x}{b(A_{55}-I_2\Omega^2)}\right\rangle_{1\times2} \tag{25}$$

The complete closed-form steady state solution for the system of flexural coupled equations is then obtained by adding the homogeneous part in equation (24) to the particular part in equation (25) to yield:

$$\{\chi(x)\}_{1\times2}=\left[\bar{G}\right]_{2\times4}\left[E(x)\right]_{4\times4}\{\bar{C}\}_{4\times1}+\{\chi_P(x)\}_{1\times2} \tag{26}$$

Exact Solution for Cantilever Composite Beam under Transverse Harmonic Forces

A cantilever composite beam subjected to (i) concentrated end harmonic forces; transverse force $\bar{P}_z(L)e^{i\Omega t}$, bending moment $\bar{M}_x(L)e^{i\Omega t}$, and (ii) distributed harmonic forces; transverse force $\bar{q}_z e^{i\Omega t}$ and bending moment $\bar{m}_x e^{i\Omega t}$ is considered as illustrated in Figure (3).

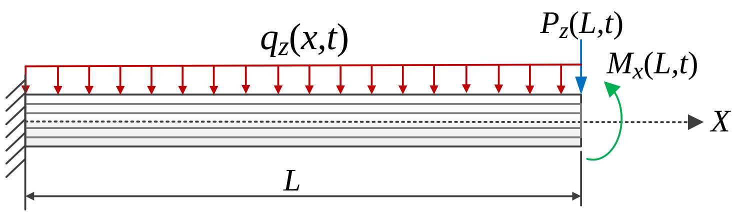

Figure 3: Composite cantilever beam under transverse harmonic forces and moments

Imposing the following cantilever boundary conditions at beam both ends, i.e., $x=0,L$:

$$\delta W(0)=0 \tag{27}$$

$$\delta\Phi_x(0)=0 \tag{28}$$

$$b\left[A_{55}\left(W'(L)+\Phi_x(L)\right)\right]=\bar{P}_z(L) \tag{29}$$

$$b\bar{D}_{11}\Phi_x'(L)=\bar{M}_x(L) \tag{30}$$

Substitution the displacement functions in equation (26) into the boundary conditions (27-30), the total closed form solution for a cantilever symmetric laminated beam becomes:

$$\{\chi_c(x)\}_{2\times1} = [\bar{G}]_{2\times4}[E(x)]_{4\times4}[\Psi_c]^{-1}_{4\times4}\{Q_c\}_{4\times1} + \{\chi_p\}_{2\times1} \tag{31}$$

where $\langle Q_c\rangle_{1\times4} = \langle -W_p \mid -\Phi_{x_p} \mid \bar{P}_z(L) - A_{55}b\Phi_{x_p} \mid \bar{M}_x(L)\rangle_{1\times4}$, and

$[\Psi_c]^T_{4\times4} = [G_i \mid 1 \mid bA_{55}e^{m_iL}(m_iG_i+1) \mid b\bar{D}_{11}m_ie^{m_iL}]^T_{4\times4}$.

Exact Solution for Simply Supported Composite Beam under Transverse Harmonic Forces

A simply supported composite beam subjected to (i) distributed harmonic forces: transverse force $\bar{q}_z e^{i\Omega t}$, bending moments $\bar{m}_x e^{i\Omega t}$, and (ii) end harmonic bending moments $\bar{M}_x(x_e)e^{i\Omega t}$ at beam both ends $(x_e = 0, L)$ is considered as shown in Figure (4).

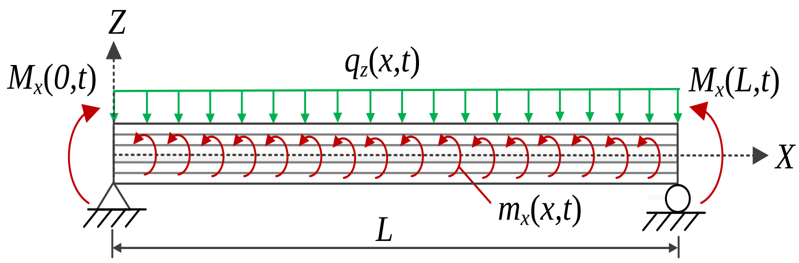

Figure 4: Simply supported composite beam under harmonic forces and moments

For simply supported beam, the boundary conditions are:

$$\delta W(0) = 0 \tag{32}$$

$$b\bar{D}_{11}\Phi'_x(0) = \bar{M}_x(0) \tag{33}$$

$$\delta W(L) = 0 \tag{34}$$

$$b\bar{D}_{11}\Phi'_x(L) = -\bar{M}_x(L) \tag{35}$$

From equation (26), by substituting into the boundary conditions (32-35), the general analytical solution for simply-supported composite beam having symmetric laminates is obtained as:

$$\{\chi_s(x)\}_{2\times1} = [\bar{G}]_{2\times4}[E(x)]_{4\times4}[\Psi_S]^{-1}_{4\times4}\{Q_s\}_{4\times1} + \{\chi_p\}_{2\times1} \tag{36}$$

where

$\langle Q_s\rangle_{1\times4} = \langle -W_p \mid \bar{M}_x(0) \mid -W_p \mid -\bar{M}_x(L)\rangle_{1\times4}$, and

$[\Psi_S]^T_{4\times4} = [G_i \mid bm_i\bar{D}_{11} \mid G_ie^{m_iL} \mid bm_ie^{m_iL}\bar{D}_{11}]^T_{4\times4}$.

NUMERICAL EXAMPLES AND DISCUSSION

While the analytical closed-form solution developed in the present study provides the steady state dynamic flexural response of symmetric laminated composite beams under various transverse harmonic forces, it can also approach the quasi-static flexural response under the given harmonic forces when adopting a very low exciting frequency

$\Omega \approx 0.01\omega_1$ compared to the first flexural natural frequency ω_1 of the system. To validate the accuracy and applicability of the present analytical solution, three examples are presented for cantilever and simply-supported composite beams with symmetrical cross-ply and angle-ply laminates. The results based on the present formulation are compared with available exact solutions in the literature and established Abaqus finite shell element. In Abaqus shell model, the shell S4R element has six degrees of freedom at each node (i.e., three translations and three rotations). The S4R shell element captures the transverse shear deformation and distortional effects.

Example 1 - Symmetric Laminated Beam under Distributed Transverse Harmonic Force

This example is presented in order to demonstrate the accuracy and validity of the present analytical solution to capture the quasi-static and to predict the natural frequencies and mode shapes of the composite symmetric laminated beams under transverse harmonic forces. The quasi-static response of the composite beam under harmonic forces is captured by using very low exciting frequency ($\Omega \approx 0.01\omega_1$) compared to the first natural frequency ω_1 of the system. For comparison, three-layered symmetric cross-ply $(0^o, 90^o, 0^o)$ laminated composite beams for both clamped-free and simply supported boundary conditions under a uniformly distributed transverse harmonic force $q_z(x,t) = 200 e^{i\Omega t} N/m$ are considered.

Quasi-static Response Validation

For static response, the three laminates have the same thickness and made of the orthotropic composite material properties: $E_{11} = 25E_{22}$, $G_{12} = G_{13} = 0.5E_{22}$, $G_{23} = 0.2E_{22}$, $\upsilon_{12} = 0.25$ and $\rho = 1350 kg/m^3$. For the comparison purpose, the transverse displacement function W for symmetric cross-ply laminated beam based on the present analytical solution are given in the following non-dimensional form [10] as: $\bar{W} = 100\, bh^3 E_{22} W \big/ q_z L^4$ and are compared with exact static solutions given by [10, 13] and [14]. Table (1) provides the non-dimensional mid-span transverse displacements $\bar{W}(x = L/2)$ for clamped and simply supported symmetric composite beams under distributed transverse forces for different span-to height ratio of (*L/h*)= 5,10,20 and 50. It is noted that the results obtained by the present formulation are in excellent agreement with results based on other exact solutions.

Table 1: Static results for symmetric cross-ply beam under distributed harmonic force

Beam type	Reference	$\bar{W}(x = L/2)$ (in mm)			
		(*L/h*)=5	**(*L/h*)=10**	**(*L/h*)=20**	**(*L/h*)=50**
Cantilever	[10]	6.703	3.328	2.485	2.248
	[13]	6.693	3.321	-	2.242
	[14]	6.698	3.323	-	2.243
	Present	6.700	3.324	2.479	2.243
Simply-supported	[10]	2.148	1.023	0.742	0.663
	[13]	2.145	1.020	-	0.660
	[14]	2.146	1.021	-	0.661
	Present	2.147	1.022	0.740	0.661

Flexural Natural Frequencies Validation

The fundamental natural frequencies of symmetric cross-ply $(0^o, 90^o, 0^o)$ laminated beams having clamped-free and simply supported boundary conditions are investigated. The steady state analyses of the composite beams under the given harmonic transverse force $q_z(\mathrm{x},t) = 200 e^{i\Omega t} N/m$ are solved in order to extract the fundamental transverse natural frequency of the given beam. For comparison, the following orthotropic composite material properties used are given [10] as: $E_{11} = 40E_{22}$, $G_{12} = G_{13} = 0.60E_{22}$, $G_{23} = 0.50E_{22}$, $\upsilon_{12} = 0.25$, $\rho = 1389 kg/m^3$. The non-dimensional natural frequencies $\bar{\omega}$ extracted from the transverse steady state responses presented in Table (2) are conducted based on the present closed form solution, and other results available in the literature survey.

Table 2: Non-dimensional fundamental natural frequencies for symmetric $(0^o, 90^o, 0^o)$ beam

Beam type	Reference	*L/h*		
		5	**10**	**20**
Cantilever	[1]	4.234	5.495	-
	[10]	4.248	5.493	6.063
	[15]	4.230	5.491	-
	Present	4.217	5.479	6.072
Simply-supported	[1]	9.208	13.61	-
	[10]	9.294	13.62	16.33
	[15]	9.207	13.61	-
	Present	9.202	13.64	16.36

Table (2) shows the first non-dimensional natural frequencies $\bar{\omega}$ for span-to-height ratio (*L/h*)= 5, 10 and 20, where the non-dimensional form is defined by: $\bar{\omega} = (\omega L^2/h)\sqrt{\rho/E_{22}}$. It is noted that the results predicted by the present analytical solution are in excellent agreement with the corresponding results obtained from the exact solution of [1] and finite element solutions of [10] and [15].

Example 2 - Simply-Supported Beam under Distributed Harmonic Transverse Force

A 6000mm span simply-supported composite beam having four-layered symmetric angle-ply $(0^o, +45^o, +45^o, 0^o)$ laminates subjected to uniformly distributed transverse harmonic force $q_z(x,t) = 24.0 e^{i\Omega t} kN/m$ is considered as shown in Figure (5). The beam cross-section has width $b = 200mm$ and thickness $h = 240mm$. The composite material properties used (taken from Jun et al. 2008), are given as: $E_{11} = 241.5GPa$, $E_{22} = 18.98GPa$, $G_{12} = G_{13} = 5.180GPa$, $G_{23} = 3.45GPa$, $\upsilon_{12} = 0.24$ and $\rho = 2015 kg/m^3$. It is required to (a) conduct a quasi-static analysis by using very low exciting frequency $\Omega \approx 0.01\omega_1$ related to the first natural frequency, (b) determine the steady state response at exciting frequency $\Omega = 1.68\omega_1$, where the first natural frequency for the beam is

$\omega_1 = 30.31 Hz$, and (c) study the effects of fiber orientation on both quasi-static and steady state dynamic responses at $\Omega = 1.32\omega_1$.

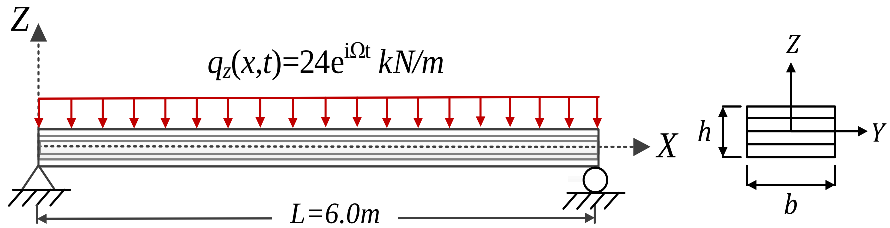

Figure (5): Simply supported laminated beam under distributed harmonic force

The present static and dynamic results are compared with Abaqus shell model solution. In Abaqus model, a total of 960 S4R shell elements are used (i.e., 6 elements along the width and 160 elements along the longitudinal axis of the beam.

Static Flexural Response

Based on the present formulation (i.e., equation 36), the static response of the simply-supported symmetric angle-ply $(0^o, +45^o, +45^o, 0^o)$ laminated beam under given distributed harmonic transverse force with very low exciting frequency $\Omega \approx 0.01\omega_1 = 0.3031 Hz$ is approached. The quasi-static results for the maximum transverse displacement W_{max} at the mid-span of the beam and related bending rotation angle Φ_{xmax} at $x = 0, L$ are presented in Table (3). It is noted that, the static response results based on the present analytical formulation are in excellent agreement with those of the Abaqus finite element model.

Table 3: Static and dynamic results for simply-supported symmetric laminated $(0^o, +45^o, +45^o, 0^o)$ beam under transverse harmonic force

Response	Variable	Abaqus Solution [1]	Present Solution [2]	%Difference =[1-2]/1
Static $\Omega = 0.01\omega_1$	W_{max} (mm)	-8.580	-8.496	0.98%
	$\Phi_{x_{max}}$ (10^{-3}rad)	4.263	4.198	1.52%
Steady state $\Omega = 1.68\omega_1$	W_{max} (mm)	4.813	4.880	-1.39%
	$\Phi_{x_{max}}$ (10^{-3}rad)	-2.248	-2.271	-1.02%

Dynamic Flexural Response

Under uniformly distributed transverse harmonic force $q_z(x,t) = 24.0\, e^{i\Omega t}\, kN/m$ with $\Omega = 1.68\omega_1 = 50.92\, Hz$, the maximum amplitudes of the transverse displacement $\overline{W}_{max}$ and related bending rotation $\overline{\Phi}_{x_{max}}$ for the steady state response of simply-supported symmetric $(0^o, +45^o, +45^o, 0^o)$ angle-ply laminated beam are provided in Table (3).

Figure (6a-b) and Figure (6c-d) show transverse displacement $W(x)$ and bending rotation $\Phi_x(x)$ along the beam span for quasi-static and steady state responses, respectively. The results given in tabular (Table 1) and graphical forms show an excellent agreement between the predictions of the bending response results based on the present solution and the results of Abaqus shell model.

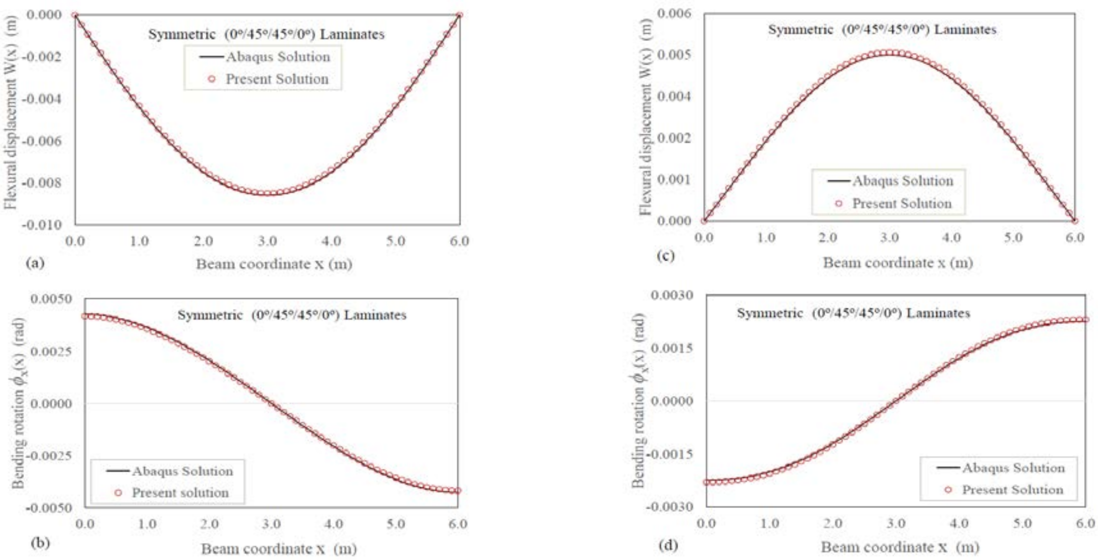

Figure 6: Quasi-static and steady state responses for simply-supported symmetric $(0^o, +45^o, +45^o, 0^o)$ laminated beam

Effect of Fiber Orientation on Composite Beam Deformations

The effects of fiber orientation on the static and steady state dynamic bending responses of simply-supported symmetric composite beams under distributed transverse harmonic force are considered. The analyses are performed for quasi-static at $\Omega \approx 0.01\omega_1 = 0.3031 Hz$ and steady state dynamic response at exciting frequency $\Omega = 1.32\omega_1 = 40.0 Hz$ for composite beam having four-layered symmetrically $(0^o/+\beta/+\beta/0^o)$ angle-ply laminates in which the outer layers are kept at 0° while the fiber angle β of the inner layers are increased ranging from $\beta = 0^o$ to 90^o in increments of 15^o. Table (4) provides the maximum transverse displacement at the mid-span of simply supported symmetric $(0^o/+\beta)_s$ angle-ply laminated beam for static and steady state responses. As a general observation, the present analytical solution yields results in very good agreements with the corresponding results given by Abaqus shell model solution. The % difference between the two solutions (given in the last column of Table 4) are arising from the discretization errors commonly used in the Abaqus finite elements.

It is noted that, as the fiber angle β increases, the maximum transverse deflections are increased for static response and decreased for the case of steady state dynamic response. It can be remarked that the laminate lay-up and stacking sequence plays the most significant role in determining the dynamic response of the composite symmetric laminated beams.

TABLE 4: Static and dynamic responses for symmetric simply supported beam of different stacking sequence laminates $(0^o/+\beta)_s$

Layup (β)	Maximum flexural deflection W_{max} (mm)					
	Static Response $\Omega \approx 0.01\omega_1$		Steady State Response $\Omega = 1.32\omega_1$		% Difference = [1-2]/1	
	Abaqus [1]	Present [2]	Abaqus [1]	Present [2]	Static	Steady State
0°	-7.719	-7.767	13.81	13.86	-0.62%	-0.36%
15°	-7.975	-7.862	13.09	13.41	1.42%	-2.44%
30°	-8.353	-8.217	12.18	12.46	1.63%	-2.30%
45°	-8.599	-8.396	11.70	12.02	2.36%	-2.74%
60°	-8.737	-8.633	11.45	11.71	1.19%	-2.27%
75°	-8.803	-8.772	11.35	11.56	0.35%	-1.85%
90°	-8.817	-8.815	11.32	11.41	0.02%	-0.80%

Example 3 - Cantilever Symmetric Laminated Beam under End Transverse Harmonic Force

A composite one end cantilever symmetric laminated beam with the same orthotropic composite material properties as that given in Example 2 subjected to concentrated transverse harmonic force $P_z(L,t) = 8.20\,e^{i\Omega t}\,kN$ applied at the free end of the cantilever beam is considered as shown in Figure (7). The length of the cantilever beam is 4000mm, while the width and thickness of the rectangular cross-section are assumed equal $b = h = 0.2794m$. The dynamic analyses are performed for cantilever beam having four-layered symmetric cross-ply $(90^o, 0^o, 0^o, 90^o)$ laminates. It is required to extract the natural frequencies and steady state bending modes. Abaqus model solution based on S4R shell element (i.e., 6 elements along the width and 120 elements along the longitudinal axis of the beam) are presented for comparison.

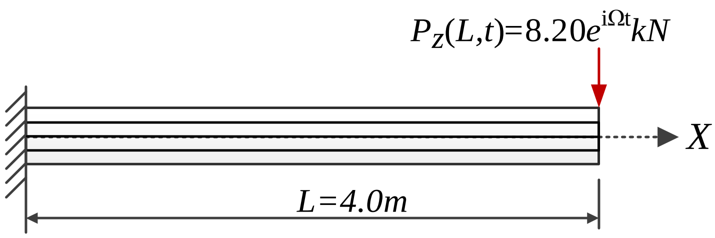

Figure 7: Cantilever symmetric laminated beam under end transverse harmonic force

Extracting Bending Natural Frequencies

Under the given harmonic force $P_z(L,t) = 8.20\,e^{i\Omega t}\,kN$, the first four natural frequencies related to the bending response are extracted from the steady state dynamic analysis when the exciting frequency Ω is varied from nearly zero to 400*Hz*. Figure (8) exhibits the four peaks which correspond to the first four bending natural frequencies of the cantilever beam in Table 4 for the present analytical and Abaqus shell model solutions. For the lower frequencies, both solutions closely predict the location of peaks associated with the first three natural frequencies. For higher frequencies, some discrepancy in the location of the peaks are observed between the two solutions. The present analytical solution yields slightly higher values than those based on the Abaqus shell model. The frequencies predicted by the present solution differed from 0.22% to 1.94% from those based on Abaqus model.

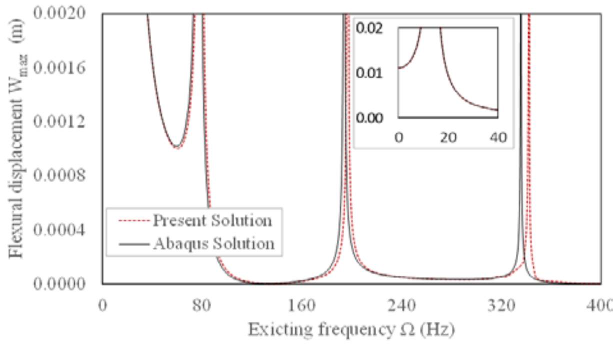

Figure 8: Natural frequencies for a cantilever symmetric $(90^o, 0^o, 0^o, 90^o)$ laminated beam

Table 4: First four natural frequencies for cantilever symmetric $(90^o, 0^o, 0^o, 90^o)$ laminated beam under end transverse harmonic force

Flexural Mode	Bending natural frequencies (*Hz*)		
	Abaqus Solution [1]	Present Solution [2]	%Difference=[1-2]/1
1	13.41	13.44	-0.22%
2	77.46	78.30	-1.08%
3	193.9	196.9	-1.55%
4	335.8	342.3	-1.94%

Steady State Bending Modes

The dynamic responses of a cantilever symmetric cross-ply laminated $(90^o, 0^o, 0^o, 90^o)$ composite beam are investigated for different values of frequency ratios Ω_i/ω_1, i.e., applied load frequency Ω_i to the first natural frequency ω_1. Figure (9) presents the first five steady state bending modes of the cantilever subjected to the given concentrated harmonic transverse force for five values of frequency ratios $\Omega_1 = 0.50\,\omega_1$, $\Omega_2 = 4.50\,\omega_1$, $\Omega_3 = 7.50\omega_1$, $\Omega_4 = 20.50\omega_1$ and $\Omega_5 = 27.50\omega_1$, respectively.

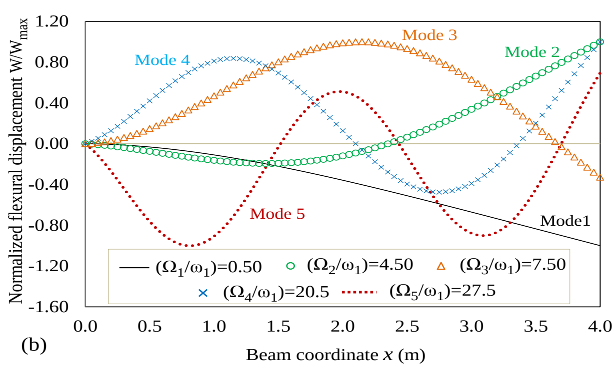

Figure 9: Natural frequencies and bending modes for a cantilever symmetric $(90^o, 0^o, 0^o, 90^o)$ laminated composite beam

CONCLUSION

From the results obtained in this study, it can be concluded that:

- Based on the first order shear deformation theory, the dynamic equations of motion for flexural vibration and related boundary conditions for composite symmetric laminated beams subjected to various transverse harmonic forces are derived via Hamilton's variational principle.
- Exact expressions for closed-form solutions of bending equations are obtained for cantilever and simply supported symmetric composite beams.
- The present analytical solutions are efficient in capturing the quasi-static and steady state dynamic responses of composite symmetric cross-ply and angle-ply laminated beams under different transverse harmonic forces. It is also capable of extracting the natural frequencies and steady state bending modes.
- Comparisons with established Abaqus finite element shell solution and exact solutions available in the literature demonstrate the validity and accuracy of the present analytical solution.

REFERENCES

[1] Khdeir, A. A. and Reddy, J. N., "Free vibration of cross-ply laminated beams with arbitrary boundary conditions", International Journal of Engineering Science, 32 (1994), 1971-1980.

[2] Yildirim, V., Sancaktar, E., Kiral, E., "Free vibration analysis of symmetric cross-ply laminated composite beams with the help of the transfer matrix approach", Communications in Numerical Methods in Engineering, 15 (9), (1999), 651-660.

[3] Banerjee, J. R., "Frequency equation and mode shape formulae for composite Timoshenko beams", Composite Structures, 51 (2001), 381-388.

[4] Banerjee, J. R., "Explicit analytical expressions for frequency equation and mode shape of composite Timoshenko beams", International Journal of Solids and Structures, 38 (2001), 2415-2426.

[5] Lijuan, F., et al., "Free vibration analysis of laminated composite beams using differential quadrature method", Tsinghua Science and Technology, 7 (6), (2002), 567-573.

[6] Tahani, M., "Analysis of laminated composite beams using layerwise displacement theories", Composite Structures, 79 (2007), 535-347.

[7] Jun, L., Hongxing, H. and Rongying, S., "Dynamic finite element method for generally laminated composite beams", 50 (2008), 466-480.

[8] Jun, L. and Hongxing, H., "Dynamic stiffness analysis of laminated composite beams using trigonometric shear deformation theory", Composite Structures, 89 (2009), 433-442.

[9] Jun, L., and Hongxing, H., "Free vibration analysis of axially loaded composite beams based on higher-order shear deformation", Meccanica, 46 (2011), 1299-1317.

[10] Vo, T., and Inam, F., "Vibration and Buckling of Cross-Ply Composite Beams using Refined Shear Deformation Theory", 2nd International Conference on Advanced Composite Materials and Technologies for Aerospace Applications, (2012), Glyndŵr University.

[11] Elshafei, M. A., "FE Modeling and Analysis of Isotropic and Orthotropic Beams Using First Order Shear Deformation Theory", Materials Sciences and Applications, 4 (2013), 77-102.
[12] Jones, R. M., "Mechanics of Composite materials", 1975, McGraw-Hill, New York.
[13] Chakraborty, A. D., et al., "Finite element analysis of free vibration and wave propagation in asymmetric composite beams with structural discontinuities", Composite Structures, 55 (1), (2002), 23-36.
[14] Khdeir, A. and Reddy, J. N., "An exact solution for the bending of thin and thick cross-ply laminated beams", Composite Structures, 37(1997), 195-203.
[15] Murthy, M. V. et al., "A refined higher order finite element foe asymmetric composite beams", Composite Structures, 67 (2005), 27-35.

NOMENCLATURES

A	Cross-sectional area
A_{55}	Coefficient
b	Width of the beam cross-section
$\bar{D}_{11}$	Bending stiffness constant
E	Modulus of elasticity
h	Height of the beam cross-section
G	Shear modulus
I_1, I_2	Densities of the composite material
J	Torsional constant
L	Length of the composite beam
$M_x(x,t)$	Concentrated harmonic moment
$m_x(x,t)$	Distributed harmonic moment
$P_z(x,t)$	Concentrated harmonic transverse force
$q_z(x,t)$	Distributed harmonic transverse force
t	Time in seconds
t_1, t_2	Time intervals
$\bar{T}$	Kinetic energy
u_p, v_p, w_p	Displacements of a point p on the cross- section along Z, Y, X axes
$\bar{U}$	Internal strain energy
$\bar{V}$	Work done by applied forces
X, Y, Z	Right-handed Cartesian coordinate system
z	Height
ρ	Density of the composite beam material
$\phi_x(x,t)$	Rotation of the normal to the mid-plane about Y axis
Ω	Exciting frequency
δ	First variation